Control Systems
Unit-V

State Space Analysis of Continuous Systems

Introduction

- Analysis and design of feedback systems like root locus and frequency response methods require the physical system to be modeled in the form of a transfer function
- An order of differential describing a physical system can be reduced to a set of first order differential equation and is represented in vector matrix notation termed as state variable model

Advantages of State Space Analysis

- It can be applied to linear and nonlinear, time invariant and time varying, single input single output system and multivariable systems
- Since it is a time domain approach, state variable model tends itself more readily to computer solution and analyses
- State variable approach enables the designer to include initial condition of the system

State

- For a dynamic system, state is defined as the smallest set of variables that must be known at any given instant
- Here, for any specified input, the future response of the system may be calculated from the given dynamic equation of state
- In dealing with linear time invariant systems, the reference time t0 will be chosen as zero

State Variables

Definition

- If at least 'n' variables $x_1(t)$, $x_2(t)$,.........., $x_n(t)$ are needed to completely describe the behavior of a dynamic system (such that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such 'n' variables $x_1(t)$, $x_2(t)$,.......,$x_n(t)$ are a set of variables termed as state variables

State Vector

- For any given system, if 'n' state variables are required to describe the complete behavior of the system, those 'n' variables can be treated as 'n' components of a vector $x(t)$. Hence, such vector is called as state vector
- A state vector is thus, a vector which determines uniquely the system state $x(t)$ for any $t \geq t_0$, one the input $u(t)$ for $t \geq t_0$ is specified

State Space

- The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ….. x_n axis is called a state space
- Any state can be represented by a point in the state space

Vector Matrix Representation of State Equation

- State equation $\frac{dx(t)}{dt} = Ax(t) + Bu(t) + Ew(t)$, Output equation y(t) = Cx(t) + Du(t) + Hw(t)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (n \times n)$$

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ d_{21} & d_{22} & \cdots & d_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ d_{q1} & d_{q2} & \cdots & d_{qp} \end{bmatrix} \quad (q \times p)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \quad (n \times p)$$

$$E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1v} \\ e_{21} & e_{22} & \cdots & e_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nv} \end{bmatrix} \quad (n \times v)$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qn} \end{bmatrix} \quad (q \times n)$$

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1v} \\ h_{21} & h_{22} & \cdots & h_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ h_{q1} & h_{q2} & \cdots & h_{qv} \end{bmatrix} \quad (q \times v)$$

State-Transition Matrix

- Let Φ(t) be the n x n matrix that represent the state-transition matrix; then it must satisfy the equation

$$\frac{d\phi(t)}{dt} = A\,\phi(t)$$

- Expression for the state-transition matrix is

$$\phi(t) = e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + ...$$

Significance of the State-Transition Matrix

- The state-transition matrix satisfies the homogeneous state equation, and it represents the free response of the system
- The state-transition matrix $\Phi(t)$ completely defines the transition of the states from the initial time $t = 0$ to any time 't' when the inputs are zero

Properties of the State-Transition Matrix

1. $\Phi(0) = I$ (the identity matrix)
2. $\Phi^{-1}(t) = \Phi(-t)$
3. $\Phi(t_2 - t_1)\,\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$ for any t_0, t_1, t_2
4. $[\Phi(t)]^k = \Phi(kt) = \Phi(t_2 - t_0)$ for $k =$ positive integer

Derivation of State Model from Block Diagram

- The state model of linear time-invariant system is thus given by

  state equation $\dot{x}(t) = Ax(t) + Bu(t)$

  output equation $y(t) = Cx(t) + Du(t)$

- For a simple mechanical system

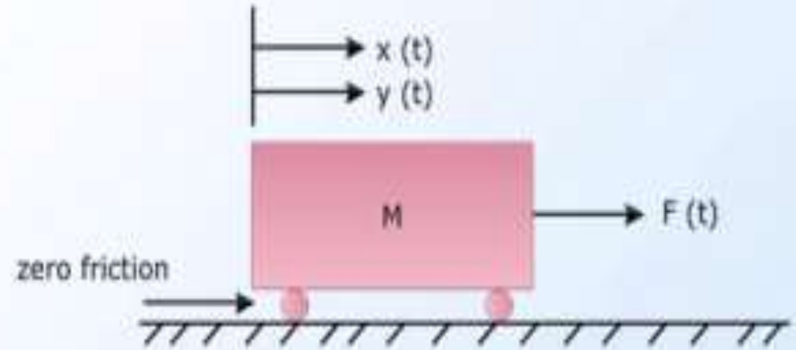
  $x_1(t) = x(t)$

  $x_2(t) = v(t)$

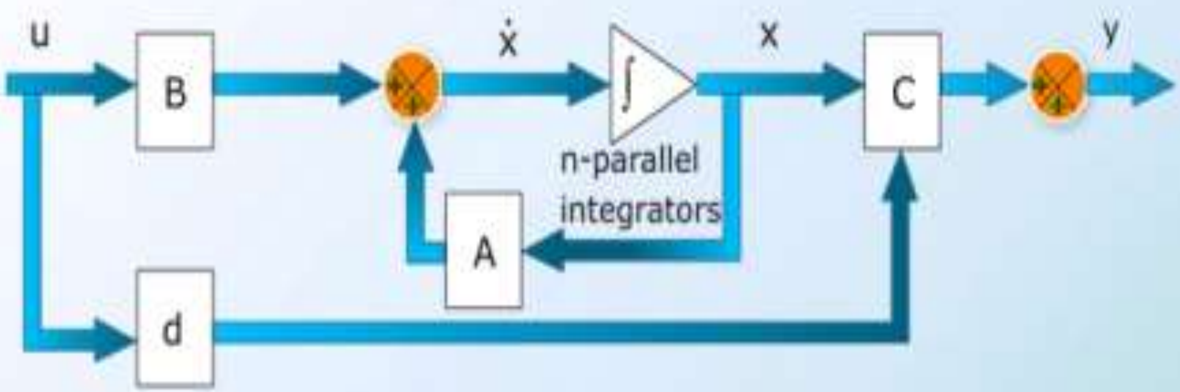
  $u(t) = F(t)$

  $$\frac{d}{dt}v(t) = \frac{1}{M}\ F(t)$$

  $$\frac{d}{dt}x(t) = v(t)$$



Simple Mechanical System



Block Diagram of State Model

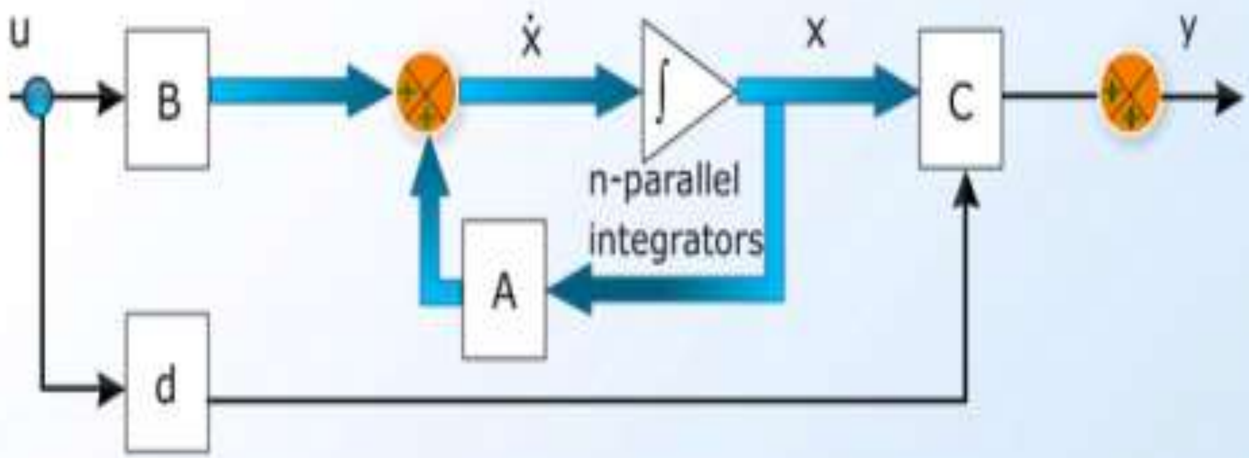
Derivation of State Model from Block Diagram

State model for Single-input-Single-output linear systems

- Let m = 1 and p = 1 in the state model of a multi-input multi-output linear system, for obtaining the following state model for a single-input-single-output linear system

$$\dot{x} = Ax + Bu$$

$$y = Cx + du$$



Block Diagram of State Model

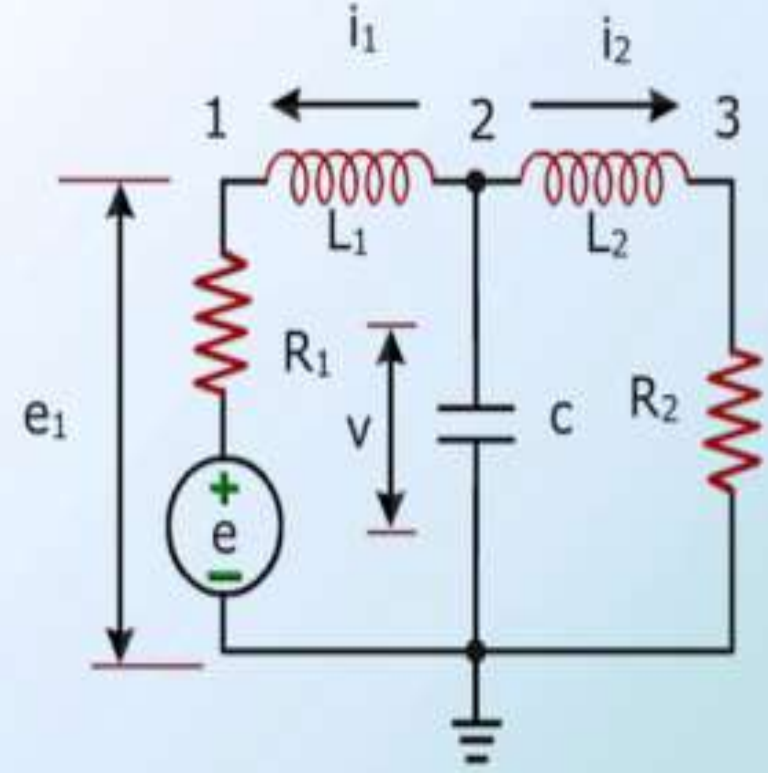
State Model for Linear Continuous-Time Systems

State-Space Representation Using Physical Variables

- It has three energy storage elements a capacitor (C) and two inductors (L1 and L2) history of the network is specified by the voltage across the capacitor and current through the inductors at t = 0

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & -1/C & -1/C\\ 1/L_1 & -R_1/L_1 & 0\\ 1/L_2 & 0 & -R_2/L_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ -1/L_1\\ 0\end{bmatrix}u$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0 & 0 & R_2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



RLC Network

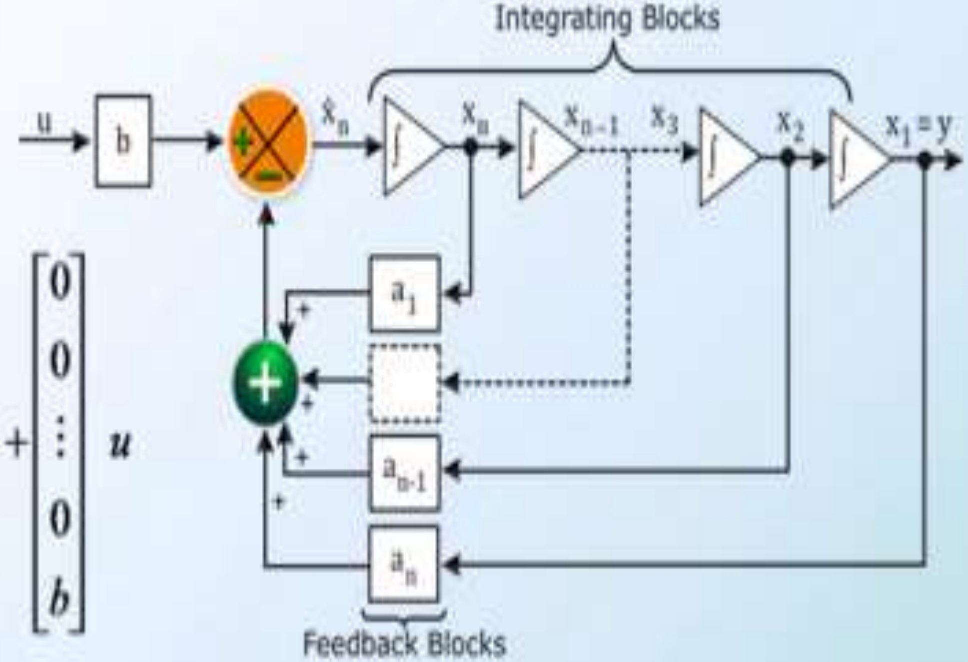
State Model for Linear Continuous-Time Systems

State-Space Representation Using Physical Variables

- If the differential equation of a state model is available, then the phase variable state model can be determined easily.
- The output y(t) to the input u(t) of a linear continuous-time system is shown

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$



Block Diagram of State Model

Diagonalization

- Some of the state models presented employed physical variables, phase variables and canonical variables
- The canonical state model wherein 'A' is in diagonal form is most suitable for this purpose
- It is therefore, useful to study techniques for transforming a general state model into a canonical one. These techniques are often referred to as Diagonalization Techniques
- $A = M^{-1} AM$ = a diagonal matrix
- The determination of the diagonalizing matrix is facilitated by the use of eigenvectors

Concept of Controllability and Observability

Controllability

- Controllability refers to the possibility in a system to transfer its initial state $x(t_0)$ to the final desired state $x(t_f)$ within a specific period of time, using a control vector $u(t)$
- In controllability, the state variables of the system are dependent on the inputs
- The condition of complete controllability can now be stated as, $\dot{z} = Jz + \widetilde{B}u$
- The elements of any row of $\widetilde{B}$ that correspond to the last row of each Jordan block are not all zero
- The system requires transformation into Jordan canonical form cause of Gilbert's method of testing controllability

Concept of Controllability and Observability

Observability

- If every state x(t0) can be completely identified by measurements of the output y(t) over a finite time interval, it is termed as completely observable
- Shielding of a system's certain state variables, makes it non-observable too

$$z = Az + \tilde{B}u$$
$$y = \tilde{C}z$$
$$= \tilde{c}_1 z_1 + \tilde{c}_2 z_2 + \ldots + \tilde{c}_n z_n$$

- If any particular $\tilde{C}$ is zero, the corresponding zi can have any value without its effect showing up in the output y.
- Thus, the necessary (it is also sufficient) condition for complete state Observability is that none of the c_i's (i.e., none of the elements of $\tilde{C} = CM$) should be zero

Derivation of Transfer Function from State Model

- There are two ways to get the transfer function from a given state model
- The first way is to draw signal flow graph from the given state model and then find the transfer function using Mason's gain formula
- Consider the state model

$$X = AX + BU$$

$$Y = CX + DU$$

- The derived equation is

$$T(s) = \frac{Y(s)}{U(s)} C(sI - A)^{-1} B + D$$

$$= C \frac{adj(sI - A)}{det(sI - A)} B + D$$